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INFINITESIMAL RIGIDITY FOR SMOOTH ACTIONS OF DISCRETE SUBGROUPS OF LIE GROUPS

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1. Introduction

Let G be a connected noncompact semisimple Lie group with finite center and no compact factors, and let $\Gamma \subset G$ be a cocompact discrete subgroup. If G has no simple factor locally isomorphic to any $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$, we established in [15], [18], [20] a rigidity property for any isometric action of Γ on a compact Riemannian manifold M. Namely, we proved that any sufficiently small perturbation of an isometric action (in the topology of pointwise convergence on Γ with the C^∞-topology on $\mathrm{Diff}(M)$) which is ergodic must also leave some smooth metric invariant. In this paper, we examine some rigidity properties of nonisometric actions of Γ. All presently known ergodic volume preserving actions of Γ on compact manifolds are of an algebraic nature (at least in the case in which G is of higher rank, i.e., all simple factors of G have real rank at least 2), and it is these algebraically defined actions which we examine. More precisely, let $\pi: G \to H$ be a homomorphism of G into another Lie group H, and let $\Lambda \subset H$ be a cocompact discrete subgroup. Then Γ acts on $M = H/\Lambda$ via π. Under mild assumptions this action will be ergodic (see [6, Chapter 2]). Unless π is trivial, there is no Γ-invariant Riemannian metric on H/Λ.

We recall that an action of a group Γ on a manifold M is called infinitesimally rigid if $H^1(\Gamma, \mathrm{Vect}(M)) = 0$, where $\mathrm{Vect}(M)$ is the space of smooth vector fields on M. This definition is of course motivated by the fact that $\mathrm{Vect}(M)$ is the Lie algebra of the infinite dimensional group $\mathrm{Diff}(M)$, and the natural action of Γ on $\mathrm{Vect}(M)$ is simply the composition of the action $\Gamma \to \mathrm{Diff}(M)$ with the adjoint representation of $\mathrm{Diff}(M)$ on its Lie algebra. We shall call an action L^2-infinitesimally rigid if the map $H^1(\Gamma, \mathrm{Vect}(M)) \to H^1(\Gamma, \mathrm{Vect}_2(M))$ is zero, where $\mathrm{Vect}_2(M)$ is the space of L^2-vector fields on M. We can then state our main results as follows.

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Theorem 4.6. *Let $\pi: G \to H$ be a homomorphism, $\Lambda \subset H$ a cocompact lattice, and let Γ act on $M = H/\Lambda$. We assume that G has no simple factor locally isomorphic to any* $\mathrm{SO}(1,n)$ *or* $\mathrm{SU}(1,n)$. *Assume the Γ action on M is ergodic.* (*This holds, e.g., if H is semisimple, Λ is an irreducible lattice, and π is nontrivial* [16, *Chapter* 2].) *Then the Γ action on M is L^2-infinitesimally rigid.*

Theorem 5.6. *With the hypotheses of Theorem* 4.6, *assume further that H is semisimple and either:*

(a) *$\pi(\Gamma)$ is dense in H; or*

(b) *$H = H_1 \times H_2$, and $\pi(\Gamma)$ projects densely into H_1 and trivially into H_2.*

Then the Γ action on M is infinitesimally rigid.

For any homomorphism $\pi: G \to H$, where H is a Lie group and G has higher rank, the work of Weil [13], Matsushima-Murakami [6], and Raghunathan [9] shows that $H^1(\Gamma, \mathrm{Ad}_H \circ (\pi|\Gamma)) = 0$, and classical results of Weil [13] (see also [10]) show how to use this infinitesimal rigidity property and the implicit function theorem to establish local rigidity. The implicit function theorem in the infinite dimensional situation one encounters in Theorem 5.6 is of course notoriously more delicate (see [4] for example). However, it is natural to hope that further development will allow use of Nash-Moser techniques to obtain a local rigidity theorem from results along the lines of Theorem 5.6.

It seems likely that Theorem 5.6 is true for all the actions considered in Theorem 4.6. In fact, most of the proof of Theorem 5.6 works under the more general hypotheses of Theorem 4.6. We discuss what is needed to complete the argument in the general case in §6 below. The arguments of the proof of Theorem 5.6 also allow us to prove the vanishing of the first cohomology group of Γ with coefficients in the smooth sections of any natural vector bundle (in the sense of [12]) over M, where M is as in Theorem 5.6. In particular, we obtain:

Theorem 5.8. *Let Γ and M be as in Theorem* 5.6. *Then* $H^1(\Gamma, C^\infty(M)) = 0$.

Simply from the fact that Γ is a Kazhdan group and preserves a finite measure on M, it follows that $H^1(\Gamma, C^\infty(M)) \to H^1(\Gamma, L^2(M))$ is the zero map, i.e., there is an L^2-solution to the functional cohomology equation. From this point of view, Theorem 5.8 is a regularity theorem.

We now outline the remainder of the paper which will simultaneously give an indication of the techniques involved. It is of course a standard technique to convert the computation of the cohomology of Γ (at least when Γ is torsion free) with coefficients in some vector space to the computation of the de Rham cohomology of $\Gamma\backslash X$ with coefficients in an associated flat vector bundle, where $X = G/K$ is the symmetric space associated to G. In §2, we develop this within our context, showing that $H^*(\Gamma, \mathrm{Vect}(M))$ can be computed via a de Rham complex of vector bundle valued forms along the leaves of a foliation. In the classical case considered by Weil [13] and Matsushima-Murakami [6], one then uses the Hodge decomposition for the Laplacian on the de Rham complex to reduce the computation to the computation of the spaces of harmonic forms. The Hodge decomposition has two aspects. The first is formal and does not depend upon ellipticity of the Laplacian. The second is the benefits that accrue from ellipticity. In our situation, the relevant Laplacian will act only in the leaf direction of the foliation (and will be elliptic along these leaves), and hence will not be elliptic as an operator on the manifold in question. Nevertheless, we do have a formal Hodge decomposition, and we will spell this out, together with some of the relevant analysis involved, in §3. In §4, we apply the local computations of Matsushima-Murakami for the Laplace operator to our situation. In particular, Lie algebra results of Raghunathan [9] can be applied to compute the relevant harmonic forms, and this leads to a proof of Theorem 4.6. §5 completes the proof of Theorem 5.6, by first proving the vanishing of the first cohomology of Γ in suitable Sobolev spaces of vector fields. The basic issue that requires understanding, given the results in §4, is the relation of the cohomology of Γ with coefficients in these Sobolev spaces to the cohomology with coefficients in the space of L^2-sections of various jet bundles over M. This is carried out in §5 under the hypotheses of Theorem 5.6. In §6 we formally set forth the type of information that would complete a proof of infinitesimal rigidity for other known actions of Γ. In §7 we consider the condition $H^1(\Gamma, C^\infty(M)) = 0$, and indicate how establishing such a result for general measure-preserving actions of Γ on compact manifolds, combined with the techniques and results in [20], would prove the main conjectures of [20].

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2. Preliminaries

We begin by establishing notation. We let G be a connected Lie group, and $K \subset G$ a maximal compact subgroup. Let $\Gamma \subset G$ be a cocompact torsion-free lattice. Let $X = G/K$, so that X is diffeomorphic to a Euclidean space, and $Y = \Gamma\backslash G/K$, which is a compact manifold with $\pi_1(Y) \cong \Gamma$. (The condition that Γ be torsion free is, as usual in many such matters, a technical convenience. As described in §4, our main results are all true without this condition.) Let M be a manifold with a volume density μ. This defines a smooth measure on M, which we also denote by μ. If $E \to M$ is a (finite dimensional) vector bundle, we let $C^k(M;E)$ denote the C^k-sections of this bundle, for $k = \infty, 0, 1, \cdots$, and if M is compact, $L^2(M;E)$ will denote the L^2-sections (with respect to the measure μ). Fixing a Riemannian metric on E then determines a Hilbert space structure on $L^2(M;E)$. As usual, $C^k(M)$ will denote the space of C^k real valued functions on M, and we let $\mathrm{Vect}^k(M) = C^k(M;TM)$ be the space of C^k vector fields. If $E \to M$ is a vector bundle, $J^kE \to M$ will denote the vector bundle of k-jets of sections of E; if E is the trivial bundle $M \times \mathbf{R}$, we set $J^k(E) = J^k(M;\mathbf{R})$. As usual, we let $j^k\colon C^\infty(M;E) \to C^\infty(M;J^kE)$ be the k-jet map. We denote the space of $\mathbf{R}$-valued p-forms on M by $A^p(M)$, and if $E \to M$ is a vector bundle, $A^p(M;E)$ denotes the space of E-valued p-forms on M, i.e., $A^p(M;E) = C^\infty(M;\mathrm{Alt}^p(TM,E))$, where Alt^p denotes a bundle of alternating p-linear maps. If E has the structure of a flat vector bundle (i.e., E admits a reduction of structure group to a countable subgroup of the relevant general linear group), then we have a natural exterior derivative $d\colon A^p(M;E) \to A^{p+1}(M;E)$ with $d^2 = 0$. We denote the corresponding cohomology group by $H^*_{\mathrm{DR}}(M;E)$. If a group H acts smoothly on M, and this action is covered by an action of H on E by vector bundle automorphisms, then all of the above spaces of sections become H-modules. If M is compact and there is an H-invariant smooth measure on M and an H-invariant Riemannian metric on E, then the action on $L^2(M;E)$ is unitary. If H preserves a flat structure on E, then the operator d on $A^*(M;E)$ commutes with H.

If T is any left Γ-space, we can form the associated bundle to the principal Γ-bundle $G/K = X \to Y = \Gamma\backslash G/K$. We denote this bundle by B_T and the projection $B_T \to Y$ by π_T. Thus, $B_T = \Gamma\backslash(T \times X)$, where Γ acts on $T \times X$ by $\gamma \cdot (t,x) = (\gamma t, \gamma x)$. The fiber of this bundle is of course T, and the images of sets of the form $\{t\} \times X$ in B_T yield a foliation $\mathscr{F}_T$ of B_T which is transverse to the fibers. We call $(B_T, \mathscr{F}_T)$ the foliated bundle associated to the Γ action on T. If $E \to M$ is a vector bundle with fiber V on which Γ acts by vector bundle automorphisms, then $B_E \to B_M$ will be a vector bundle with fiber V. If Γ preserves a flat structure on $E \to M$, then $B_E \to B_M$ inherits a flat structure. We remark, however, that even if E has no flat structure, we have a natural notion of a section of $B_E \to B_M$ being locally constant along the leaves of $\mathscr{F}_M$.

We shall need another closely related and standard construction, namely that of induced action. Thus, we let $I_T = \Gamma\backslash(T \times G)$ so that G acts on the right of I_T by $\Gamma(t, h)g = \Gamma(t, hg)$. The stabilizers in G of points in I_T are all conjugates in G of stabilizers in Γ of points in T, and in particular the action of G on I_T is locally free (i.e. has discrete stabilizers). I_T is a bundle over $\Gamma\backslash G$ with fiber T, and the natural map $\tilde{\pi}_T \colon I_T \to \Gamma\backslash G$ is a G-map. We call the action of G on I_T the action induced from the Γ-action on T. If T is compact, I_T is as well, and if Γ preserves a finite measure μ on T, G will preserve the finite measure ν on I_T given by $\nu = \int_{z\in\Gamma\backslash G}^{\oplus} \mu_z \, dm$, where μ_z is the measure on the fiber over $z \in \Gamma\backslash G$ defined by μ, and m is the G-invariant volume on $\Gamma\backslash G$. If $E \to M$ is a vector bundle with fiber V, so is $I_E \to I_M$, and if Γ preserves a flat structure or Riemannian metric on E, G will as well on I_E. We have a commuting diagram:

$$\begin{array}{ccc} I_E & \longrightarrow & I_E/K \cong B_E \\ \downarrow & & \downarrow \\ I_M & \longrightarrow & I_M/K \cong B_M \end{array}$$

The orbits of G in I_M project onto the leaves of $\mathscr{F}_M$.

A particularly important case of a flat vector bundle $E \to M$ on which Γ acts by automorphisms preserving the flat structure is $E = M \times V$, where V is a vector space, and the Γ-action is given by $\gamma(m, v) = (\gamma m, \rho(\gamma)v)$, where $\rho \colon \Gamma \to \mathrm{GL}(V)$ is a linear representation. In this case $C^\infty(M; E) \cong C^\infty(M) \otimes V$ as Γ-modules. We denote the corresponding bundle B_E by B_ρ. In this case we also have associated to ρ a flat vector bundle $E_\rho \to Y = \Gamma\backslash G/K$, and we clearly have that $B_\rho = \pi_M^*(E_\rho)$, the pull-back of E_ρ by $\pi_M \colon B_M \to Y$. Similarly, we have an associated flat vector bundle E_ρ^* on $\Gamma\backslash G$, and denoting $I_{M\times V}$ by I_ρ, we have that $I_\rho = \tilde{\pi}_M^*(E_\rho^*)$, where $\tilde{\pi}_M \colon I_M \to \Gamma\backslash G$.

If the representation ρ is actually the restriction to Γ of a representation $\rho \colon G \to \mathrm{GL}(V)$, then it is well known that $E_\rho \to Y$ can also be considered as the vector bundle associated to the principal K-bundle $\Gamma\backslash G \to Y$ and the representation $\rho|K \colon K \to \mathrm{GL}(V)$ (see [10], e.g.; we recall the proof below). We can give a similar description for B_ρ. Namely, we have the following maps:

$$\begin{array}{ccccc} & & M \times G & & \\ & \swarrow & & \searrow & \\ I_M = \Gamma\backslash(M \times G) & & & & M \times G/K \\ & \searrow & & \swarrow & \\ & & B_M = \Gamma\backslash(M \times G/K) & & \end{array}$$

The map $I_M \to B_M$ is a principal K-bundle.

Lemma 2.1. *If $\rho\colon G \to \mathrm{GL}(V)$ is a representation, then $B_\rho \to B_M$ is naturally isomorphic to the vector bundle over B_M associated to the principal K-bundle $I_M \to B_M$ and the representation $\rho|K$.*

Proof. Consider the map $\Phi\colon M \times G \times V \to M \times G \times V$ given by $\Phi(m,g,v) = (m,g,\rho(g)v)$. We let $\Gamma \times K$ act on $M \times G \times V$ in two ways: first, by $\gamma(m,g,v)k = (\gamma m, \gamma g k, \rho(k)^{-1}v)$; and second, by $\gamma \circ (m,g,v) \circ k = (\gamma m, \gamma g k, \rho(\gamma)v)$. Then under the first action the map

$$\Gamma\backslash(M \times G \times V)/K \to \Gamma\backslash(M \times G)/K$$

is exactly the projection of the associated bundle for the representation $\rho|K$, and under the second action the map

$$\Gamma\backslash(M \times G \times V)/K \to \Gamma\backslash(M \times G)/K$$

is the projection $B_\rho \to B_M$. The lemma then follows from the observation that

$$\Phi(\gamma(m,g,v)k) = \gamma \circ (\Phi(m,g,v)) \circ k.$$

As in the classical case of E_ρ, this realization of B_ρ is no longer naturally flat, but has the compensating feature that the structure group is compact.

If $E \to M$ is a vector bundle on which Γ acts by vector bundle automorphisms, we shall be interested in $H^*(\Gamma; C^\infty(M;E))$. We first remark that we can obtain a de Rham type realization of this cohomology. Namely, if $\mathscr{E} \to B_M$ is a vector bundle, we let $A^p(\mathscr{F}_M;\mathscr{E})$ be the space of $\mathscr{E}$-valued p-forms along the leaves of $\mathscr{F}_M$, i.e., $A^p(\mathscr{F}_M;\mathscr{E}) = C^\infty(B_M;\mathrm{Alt}^p(T\mathscr{F}_M,\mathscr{E}))$, where $T\mathscr{F}_M$ is the tangent bundle to $\mathscr{F}_M$. If $\mathscr{E} = B_E$, where E is a vector bundle over M on which Γ acts by vector bundle automorphisms, then we have a natural exterior derivative along the leaves of $\mathscr{F}_M$, $d_{\mathscr{F}}\colon A^p(\mathscr{F}_M;B_E) \to A^{p+1}(\mathscr{F}_M;B_E)$, with $d_{\mathscr{F}}^2 = 0$. We denote the corresponding cohomology groups by $H^*_{\mathrm{DR}}(\mathscr{F}_M;B_E)$.

Lemma 2.2. *Let $E \to M$ be a vector bundle on which Γ acts by vector bundle automorphisms. Then there is a natural isomorphism*

$$H^*(\Gamma; C^\infty(M;E)) \cong H^*_{\mathrm{DR}}(\mathscr{F}_M;B_E).$$

Proof. The action of Γ on $C^\infty(M;E)$ defines a local system on $Y = \Gamma\backslash X$, and $H^*(\Gamma; C^\infty(M;E))$ is isomorphic to the cohomology of Y with coefficients in this local system. For any open $U \subset Y$, let $\mathscr{A}(U)$ be the smooth sections of $B_E \to B_M$ defined on $\pi_M^{-1}(U)$ which are locally constant along the leaves of $\mathscr{F}_M$. It is easy to see that the sheaf of locally constant sections of the locally constant sheaf corresponding to the above local system is naturally identified with the sheaf $\mathscr{A}$ on Y. Thus we have a natural isomorphism $H^*(\Gamma; C^\infty(M;E)) \cong H^*(Y,\mathscr{A})$. For $U \subset Y$ open, and

$0 \le p \le \dim(Y)$, let $\mathscr{A}^p(U)$ be the smooth sections of $\mathrm{Alt}^p(T\mathscr{F}, B_E) \to B_M$ defined on $\pi_M^{-1}(U)$. Then $\mathscr{A}^p$ is clearly a fine sheaf on Y. Furthermore, the maps $d_{\mathscr{F}}|_{\pi_M^{-1}(U)}$ define sheaf maps $d_{\mathscr{F}}\colon \mathscr{A}^p \to \mathscr{A}^{p+1}$ and

$$0 \to \mathscr{A} \to \mathscr{A}^0 \to \cdots \to \mathscr{A}^p \to \cdots \to \mathscr{A}^{\dim(Y)} \to 0$$

is clearly a chain complex of sheaves. A simple variant of the Poincaré lemma (performed "leafwise"; cf. [7]) shows that this complex is actually a fine resolution of $\mathscr{A}$. The cohomology of $\mathscr{A}$ is thus the cohomology of the global sections of the resolution, and the lemma follows.

We shall need a more explicit version of this isomorphism in the case $E = M \times V$ and the section is given by $\rho\colon \Gamma \to \mathrm{GL}(V)$. Let $q\colon M \times X \to B_M = \Gamma\backslash(M \times X)$ be the natural quotient map, and $p_X\colon M \times X \to X$ the projection. Then $\mathrm{Alt}^p(p_X^* TX, V)$ is the bundle of V-valued p-forms on $M \times X$ in the "X-direction". We have a natural action of Γ on $C^\infty(M \times X; \mathrm{Alt}^p(p_X^* TX, V))$ and a natural isomorphism of the space of Γ-invariants, $C^\infty(M \times X; \mathrm{Alt}^p(p_X^*(TX, V))^\Gamma$, with $A^p(\mathscr{F}_M; B_\rho)$. For any $\omega \in C^\infty(M \times X, \mathrm{Alt}^p(p_X^* TX, V))$, we can consider ω as a family of V-valued p-forms on X parametrized by M, i.e. $m \to \omega_m \in A^p(X, V)$, $m \in M$. Then $d_{\mathscr{F}}\omega$ corresponds to the family $d(\omega_m)$, where d is the ordinary exterior derivative in $A^*(X, V)$. A form $\omega \in C^\infty(M \times X; \mathrm{Alt}^p(p_X^* TX, V))$ is Γ-invariant if and only if $\rho(\gamma)\gamma^*(\omega_{\gamma^{-1}m}) = \omega_m$, where we consider $\mathrm{GL}(V)$ as acting on $A^p(X, V)$ by pointwise application in the range. We observe that we can also apply both γ and d to sections of $\mathrm{Alt}^p(p_X^* TX, V)$ for which all ω_m, $m \in M$, are smooth, but for which ω_m do not necessarily vary smoothly in m.

For $\omega \in A^1(\mathscr{F}_M, B_\rho)$, with $d_{\mathscr{F}}\omega = 0$, let $\omega_m \in A^1(X, V)$ be the corresponding family of closed V-valued 1-forms on X. For $\gamma \in \Gamma$, let $\alpha_\omega(\gamma)\colon M \to V$ be given by $\alpha_\omega(\gamma) = \int_I \omega_m$, where I is a path in X from x_0 to γx_0, and $x_0 \in X$ is some point fixed in advance. Since ω_m is closed and X is simply connected, $\alpha_\omega(\gamma)$ is independent of this choice of path. It is easy to see that the Γ-invariance of the family $\{\omega_m\}$ implies that $\alpha_\omega\colon \Gamma \to C^\infty(M, V)$ is a 1-cocycle. Namely, $\alpha_\omega(\gamma_1\gamma_2)(m) = \int_{I_2 I_1} \omega_m$, where I_1 is a path from x_0 to $\gamma_1 x_0$ and I_2 is a path from $\gamma_1 x_0$ to $\gamma_1\gamma_2 x_0$. Therefore $\alpha_\omega(\gamma_1\gamma_2)(m) = \int_{I_1} \omega_m + \int_{I_2} \omega_m$, and

$$\int_{I_2} \omega_m = \int_{I_2} \rho(\gamma_1)\gamma_1^*(\omega_{\gamma_1^{-1}m}) = \rho(\gamma_1)\int_{I_2} \gamma_1^*\omega_{\gamma_1^{-1}m} = \rho(\gamma_1)\int_{\gamma_1^{-1}I_2} \omega_{\gamma_1^{-1}m}.$$

Since $\gamma_1^{-1}I_2$ is a path from x_0 to $\gamma_2 x_0$, we obtain

$$\int_{I_2} \omega_m = \rho(\gamma_1)\alpha_\omega(\gamma_2)(\gamma_1^{-1}m) = \gamma_1 \cdot \alpha_\omega(\gamma_2)$$

for the action of Γ on $C^\infty(M, V)$. This verifies that $\alpha_\omega(\gamma_1\gamma_2) = \alpha_\omega(\gamma_1) + \gamma_1\alpha_\omega(\gamma_2)$.

Suppose now that $d_{\mathscr{F}}\theta = \omega$ for $\theta \in A^0(\mathscr{F}_M; B_\rho)$. Letting $\theta_m: X \to V$ be the corresponding functions, we have $d\theta_m = \omega_m$. Thus,

$$\alpha_\omega(\gamma)(m) = \int_I \omega_m = \int_I d\theta_m = \theta_m(\gamma x_0) - \theta_m(x_0).$$

But by the invariance of $\{\theta_m\}$, we have

$$\theta_m(\gamma x_0) = \rho(\gamma)[(\gamma^*\theta_{\gamma^{-1}m})(\gamma x_0)] = \rho(\gamma)\theta_{\gamma^{-1}m}(x_0).$$

Therefore, if for $x \in X$ we let $\theta^x: M \to V$ be $\theta^x(m) = \theta_m(x)$, we have $\alpha_\omega(\gamma) = \gamma \cdot \theta^{x_0} - \theta^{x_0}$ for the action of Γ on $C^\infty(M, V)$, i.e., α_ω is a coboundary, if $\omega = d_{\mathscr{F}}\theta$. Thus, $\omega \mapsto \alpha_\omega$ induces a map $H^1(\mathscr{F}_M, B_\rho) \to H^1(\Gamma; C^\infty(M, V))$. In this case, Lemma 2.2 becomes

Lemma 2.3. *$\omega \to \alpha_\omega$ induces an isomorphism*

$$H^1(\mathscr{F}_M, B_\rho) \cong H^1(\Gamma, C^\infty(M, V)).$$

The above discussion shows that even if θ is not smooth as a function of m, but for each fixed m we still have $d\theta_m = \omega_m$, then α_ω will be a coboundary in a Γ-module consisting of functions more general than smooth ones. More precisely, we have:

Lemma 2.4. *Let $\rho: \Gamma \to \mathrm{GL}(V)$ be a linear representation. Assume M is compact. Suppose that for each $\omega \in A^1(\mathscr{F}_M, B_\rho)$, there is a measurable section θ of $B_\rho \to B_M$ with corresponding family of functions $\theta_m: X \to V$, $m \in M$, satisfying the following*:

(i) *$\theta \in L^2(B_M, B_\rho)$.*

(ii) *θ is smooth along the leaves of $\mathscr{F}_M$, or equivalently, each θ_m, $m \in M$ is smooth.*

(iii) *For a.e. $m \in M$, $d\theta_m = \omega_m$.*

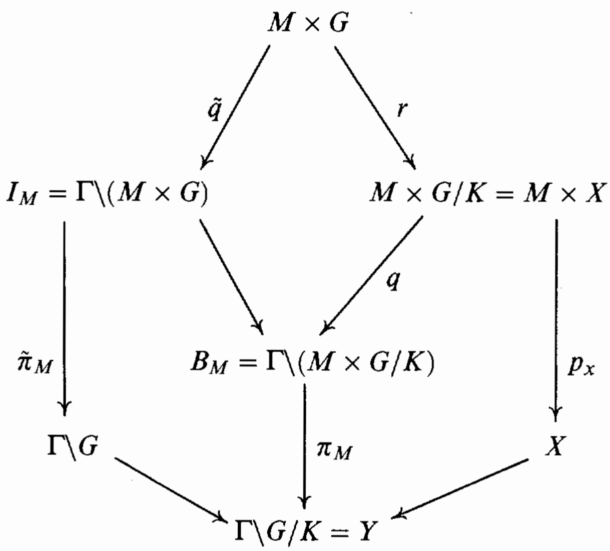
Then the map $H^1(\Gamma, C^\infty(M, V)) \to H^1(\Gamma, L^2(M, V))$ is 0.

Proof. Let $[\alpha] \in H^1(\Gamma, C^\infty(M, V))$. By Lemma 2.3, we can assume $\alpha = \alpha_\omega$ for some Γ-invariant section ω of $\mathrm{Hom}(p_X^*TX, V) \to M \times X$. Since $C^\infty(M \times X, \mathrm{Hom}(p_X^*TX, V))^\Gamma$ is naturally isomorphic to $A^1(\mathscr{F}_M, B_\rho)$ (and this isomorphism extends to sections which are only measurable), our hypotheses and Fubini's theorem imply that there is a measurable function $\theta: M \times X \to V$ such that:

(i) θ is a Γ-map;
(ii) for a.e. $x \in X$, $\theta^x: M \to V$ is in $L^2(M, V)$;
(iii) for each $m \in M$, $\theta_m(x) = \theta(m, x)$ is in $C^\infty(X, V)$;
(iv) $d\theta_m = \omega_m$.

The argument preceding Lemma 2.3 holds for any $x_0 \in X$. Choosing x_0 such that $\theta^{x_0} \in L^2(M, V)$, that argument proves the lemma.

Now assume that $\rho\colon G \to \mathrm{GL}(V)$ is a linear representation. Following the classical case for describing the exterior derivative on $A^*(\Gamma\backslash X, E_\rho)$ (which appears in [6], [8], e.g.), we now describe the complex $C^\infty(M \times X, \mathrm{Alt}^*(p_X^* TX, V))^\Gamma$ in terms of forms on I_M, i.e., in terms of K-invariant V-valued forms on $\Gamma\backslash(M \times G)$, where the K-action is given by $\Gamma(m, g)k = \Gamma(m, gk)$. To do this, we use the isomorphism given by Lemma 2.1. For ease of reference, we first recall notation for some maps and label some other ones in the following diagram:



The action of G on I_M is locally free, so we can let $\mathcal{O}$ be the orbit foliation for this action. Thus, $T\mathcal{O} = \tilde{\pi}_M^*(T(\Gamma\backslash G))$. The diffeomorphism Φ in the proof of Lemma 2.1 induces an injective map

$$\Psi\colon C^\infty(M \times X; \mathrm{Alt}^*(p_X^* TX, V))^\Gamma \to C^\infty(I_M; \mathrm{Alt}^*(T\mathcal{O}, V))^K,$$

where K acts on V via ρ. If $A \in \mathfrak{g}$, this clearly defines a vector field (which we still denote by A) on I_M with A taking values in $T\mathcal{O} \subset TI_M$ at all points in I_M. The image of Ψ is exactly $\{\eta^0 | i(A)\eta^0 = 0 \text{ for all } A \in \mathfrak{k}\}$. Here $i(A)$ is the natural map

$$C^\infty(I_M; \mathrm{Alt}^p(T\mathcal{O}, V)) \to C^\infty(I_M; \mathrm{Alt}^{p-1}(T\mathcal{O}, V)).$$

We set

$$C^\infty(I_M; \mathrm{Alt}^*(T\mathcal{O}, V))_0^K = \{\eta^0 | i(A)_\eta^0 = 0 \text{ for all } A \in \mathfrak{k}\},$$

so that Ψ is an isomorphism onto this space. The map $\Psi(\eta) = \eta^0$ is characterized by

$$\rho(g)(r^*(\eta)_{(m,g)}) = \tilde{q}^*(\eta^0).$$

On the domain of Ψ, we have the exterior derivative in the X direction (which we recall corresponds to $d_{\mathscr{F}}$). We now give a description of this differential operator carried over to $C^\infty(I_M; \mathrm{Alt}^*(T\mathscr{O}, V))_0^K$ via Ψ. We first observe that for $y \in I_M$, we have a natural identification of $T\mathscr{O}_y$ with $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where the latter is the Cartan decomposition of the Lie algebra of G. We may thus identify $C^\infty(I_M; \mathrm{Alt}^*(T\mathscr{O}, V))$ with $C^\infty(I_M; \mathrm{Alt}^*(\mathfrak{g}, V))$, and the G-action on the latter given by the G-action on I_M, the representation ρ on V, and Ad_G on $\mathfrak{g}$. By the condition $i(A)(\eta^0) = 0$ for all $A \in \mathfrak{k}$, we may clearly identify $C^\infty(I_M; \mathrm{Alt}^*(\mathfrak{g}, V))_0^K$ with $C^\infty(I_M, \mathrm{Alt}^*(\mathfrak{p}, V))^K = C^\infty(I_M, \Lambda(\mathfrak{p})^* \otimes V)^K$. Fix a basis $\{X_i\}$ of $\mathfrak{g}$ orthonormal for the Killing form, with $X_i \in \mathfrak{p}$, $1 \leq i \leq N$, and $X_i \in \mathfrak{k}$, $N+1 \leq i \leq n = \dim G$. Let $\omega_i \in \mathfrak{g}^*$ be the dual basis. For any increasing l-tuple $I = (i_1, \cdots, i_l)$, where $l \leq N$, let $\omega_I = \omega_{i_1} \wedge \cdots \wedge \omega_{i_l}$. We let $(I, \hat{j})$ denote the $(l-1)$-tuple I with i_j deleted. Then any element $\eta^0 \in C^\infty(I_M, \Lambda(\mathfrak{p})^* \otimes V)$ is given uniquely by $\eta^0 = \sum_I \eta_I \otimes \omega_I$, where the $\eta_I \colon I_M \to V$ are smooth functions. We define, for $\eta^0 \in C^\infty(I_M, \Lambda^p(\mathfrak{p})^* \otimes V)$, two elements of $C^\infty(I_M, \Lambda^{p+1}(\mathfrak{p})^* \otimes V)$. Let

$$(D\eta^0)_I = \sum_{j=1}^{p+1} (-1)^{j+1} X_{i_j}(\eta_{(I,\hat{j})}),$$

and

$$(d_\rho \eta^0)_I = \sum_{j=1}^{p+1} (-1)^{j+1} \rho(X_{i_j})(\eta_{(I,\hat{j})}).$$

Here we continue to denote by ρ the representation of $\mathfrak{g}$ induced by ρ. The computation of [8, p. 24], [6] carries over essentially verbatim, and analogous to the result for the classical case we have:

Lemma 2.5. *Under the isomorphism $\Psi(\eta) = \eta^0$, $d\eta$ corresponds to $(D + d_\rho)(\eta^0)$.*

3. A formal Hodge decomposition

The operator $d_{\mathscr{F}}$ defined in §2 is an example of an operator on a function space of a foliated manifold which is a differential operator along the leaves of the foliation. A general development of some basic properties of such operators appears in [7], in the more general setting of a metric space foliated by smooth manifolds. Rather than develop the most general theory, we shall indicate some developments in the special framework in which we shall need them. We should emphasize, however, that while we are considering differential operators along the leaves of a foliation, as are Moore-Schochet [7], much of our development is in a very different direction from that of [7]. In the first place, the smoothness of the transversal direction will ultimately play a basic role for us (see §5), while it plays no role in [7]. More immediately, however, is that the spaces on which we consider our operators to be acting are of a very different nature than those in [7]. Namely, in [7], the leaves are "lifted out" of the ambient manifold, and the differential operators act on smooth functions (or sections) which are compactly supported on the leaf, or L^2 on the leaf, etc. On the other hand, we shall be interested in the operator acting on functions (or sections) defined on the ambient manifold itself. In the generic case, these spaces will have very different behavior.

We suppose henceforth that there is a smooth Γ-invariant measure μ on M. Since $\Gamma\backslash X$ is endowed with a natural volume form, the space B_M carries an induced smooth measure ν. It is important to observe that locally, ν is a product measure. We formally record this for ease of reference.

Lemma 3.1. *If $U \subset Y = \Gamma\backslash X$ is a small open set, we have a diffeomorphism $U \times M \to \pi_M^{-1}(U)$ of bundles over U such that $\nu|_{\pi^{-1}(U)}$ corresponds to the product measure* $(\mathrm{vol}|_U) \times \mu$.

Let (ρ, V) be a finite dimensional representation of G, and $E_\rho \to \Gamma\backslash X$ the associated vector bundle. Then $\pi_M^*(E_\rho)$ is a vector bundle over B_M. Fix a K-invariant metric on V. Then (via Lemma 2.1, for example), E_ρ has a corresponding Riemannian metric and hence so does $\pi_M^*(E_\rho)$.

Let $D\colon C^\infty(\Gamma\backslash X; E_\rho) \to C^\infty(\Gamma\backslash X; E_\rho)$ be a differential operator with adjoint D^*. For any such D, define a differential operator $D_{\mathscr{F}}$ on $C^\infty(B_M; \pi_M^*(E_\rho))$ by letting $D_{\mathscr{F}}$ act as D in the leaf direction of the foliation $\mathscr{F}_M$. The exterior derivative $d_{\mathscr{F}}$ of §2 is such an example.

Lemma 3.2. $(D_{\mathscr{F}})^* = (D^*)_{\mathscr{F}}$.

Proof. It suffices to see equality when applied to a smooth f with support contained in $\pi_M^{-1}(U)$ where $U \subset \Gamma\backslash X$ is a small open set. This follows easily from Lemma 3.1.

We now define the "leaf-wise" Sobolev spaces. Let Δ be the Laplace operator on E_ρ-valued forms on $\Gamma\backslash X$. Thus, if d is the exterior derivative on E_ρ-valued forms, then $\Delta = d^*d + dd^*$. Then $\Delta_{\mathscr{F}}$ operates on $C^\infty(B_M; \mathrm{Alt}^*(T\mathscr{F}_M, B_\rho))$. The restriction of $\Delta_{\mathscr{F}}$ to a fixed leaf is elliptic, although $\Delta_{\mathscr{F}}$ is itself not elliptic (as long as M is nontrivial.) The leaves of $\mathscr{F}$ will not in general be compact. Given such a B_ρ-valued form

ω along the leaves of $\mathcal{F}_M$, define $\|\omega\|_{2,r,\mathcal{F}} = ((I+\Delta_{\mathcal{F}})^r\omega,\omega)^{1/2}$ where $(\ ,\ )$ is the ordinary L^2 inner product. (We recall that both $T\mathcal{F}_M$ and B_ρ have metrics.) We let $L^{2,r}_{\mathcal{F}}(B_M;\mathrm{Alt}^*(T\mathcal{F}_M,B_\rho))$ be the completion of $C^\infty(B_M;\mathrm{Alt}^*(T\mathcal{F}_M,B_\rho))$ with respect to $\|\ \|_{2,r,\mathcal{F}}$. We may clearly identify $L^{2,r}_{\mathcal{F}}(B_M;\mathrm{Alt}^*(T\mathcal{F}_M,B_\rho))$ with a dense subspace of the corresponding L^2-space. The following is a straightforward consequence of the Sobolev embedding theorem and Fubini's theorem (and Lemma 3.1).

Lemma 3.3. *If $f\in L^{2,r}_{\mathcal{F}}\subset L^2$, then (as an element of L^2) f can be represented by ω which is C^l on each leaf, where $l=r-\dim(\Gamma\backslash X)/2$.*

Proof. Suppose $f_j\in C^\infty$ and $f_j\to f$ in $L^{2,r}_{\mathcal{F}}$. Fix an open set $U\subset\Gamma\backslash X$ over which we can write $\pi_M^{-1}(U)\cong U\times M$. Then each f_j and f can be written on $\pi_M^{-1}(U)$ as a family of E_ρ-valued forms on U parametrized by $m\in M$, say f_j^m. Since $\{f_j\}$ is Cauchy in $\|\ \|_{2,r,\mathcal{F}}$, by Fubini we have for a.e. $m\in M$ that $\langle (I+\Delta)^r(f_j^m-f_k^m),f_j^m-f_k^m\rangle\to 0$ as $j,k\to\infty$. Since Δ is elliptic of order 2, the usual Sobolev embedding theorem implies that for a.e. $m\in M$, f_j^m converges to a C^l function. Since $f_j\to f$ in L^2, we deduce that for a.e. $m\in M$, $f|\pi_M^{-1}(U)$ is C^l on the connected component of m of the intersection of the leaf through m with $\pi_M^{-1}(U)$. Let $M_0\subset M$ be the null set for which f^m is not C^l. Then $\Gamma\cdot M_0$ will also be null. Hence $f|\pi_M^{-1}(U)$ will be C^l along a conull set of leaves. Covering $\Gamma\backslash X$ with finitely many (or even countably many) such U, we see that f will be C^l on a conull set of leaves. Letting $f=0$ on the complementary null set completes the proof.

For r even and all l, the map $(I+\Delta_{\mathcal{F}})^{r/2}$ defined on C^∞ sections extends uniquely to a unitary isomorphism

$$(I+\Delta_{\mathcal{F}})^{r/2}\colon L^{2,r+l}_{\mathcal{F}}(B_M;\mathrm{Alt}^p(T\mathcal{F}_M,B_\rho))\to L^{2,l}_{\mathcal{F}}(B_M;\mathrm{Alt}^p(T\mathcal{F}_M,B_\rho)).$$

Similarly, for each r the exterior derivative d extends to a continuous map

$$d_{\mathcal{F}}\colon L^{2,r}_{\mathcal{F}}(B_M,\mathrm{Alt}^p(T\mathcal{F},B_\rho))\to L^{2,r-1}_{\mathcal{F}}(B_M,\mathrm{Alt}^{p+1}(T\mathcal{F},B_\rho)).$$

For r even, the following diagram commutes:

$$\begin{array}{ccc}
L^{2,r+1}_{\mathcal{F}}(B_M,\mathrm{Alt}^p(T\mathcal{F},B_\rho)) & \xrightarrow{d_{\mathcal{F}}} & L^{2,r}_{\mathcal{F}}(B_M,\mathrm{Alt}^{p+1}(T\mathcal{F},B_\rho)) \\
\cong\Big\downarrow (I+\Delta_{\mathcal{F}})^{r/2} & & \cong\Big\downarrow (I+\Delta_{\mathcal{F}})^{r/2} \\
L^{2,1}_{\mathcal{F}}(B_M;Alt^p(T\mathcal{F},B_\rho)) & \xrightarrow{d_{\mathcal{F}}} & L^2(B_M,\mathrm{Alt}^{p+1}(T\mathcal{F},B_\rho))
\end{array}$$

Then a standard argument yields the following regularity result.

Lemma 3.4. *Suppose that for every smooth $\omega\in C^\infty(B_M;\mathrm{Alt}^1(T\mathcal{F}_M,B_\rho))$ with $d_{\mathcal{F}}\omega=0$ we have that $\omega=d_{\mathcal{F}}\eta$ for some $\eta\in L^{2,1}_{\mathcal{F}}(B_M;B_\rho)$. Then η is smooth along the leaves of $\mathcal{F}_M$.*

Proof. If $d_{\mathscr{F}}\omega = 0$, we also have that for each even r, $d_{\mathscr{F}}((I+\Delta_{\mathscr{F}})^{r/2}\omega) = 0$ since $d_{\mathscr{F}}$ commutes with $\Delta_{\mathscr{F}}$. Thus, we can find $\alpha \in L^{2,1}_{\mathscr{F}}(B_M; B_\rho)$ such that $d_{\mathscr{F}}\alpha = (I+\Delta_{\mathscr{F}})^{r/2}\omega$. Then $d_{\mathscr{F}}((I+\Delta_{\mathscr{F}})^{-r/2}\alpha) = \omega$, and by Lemma 3.3, $(I+\Delta_{\mathscr{F}})^{-r/2}\alpha$ is C^l along the leaves. If η_1, η_2 are measurable sections of $B_\rho \to B_M$ which are C^1 along the leaves and $d_{\mathscr{F}}\eta_1 = d_{\mathscr{F}}\eta_2$, then $d_{\mathscr{F}}(\eta_1 - \eta_2) = 0$, so $\eta_1 - \eta_2$ is locally constant along the leaves. Thus, if one is C^l along the leaves, so is the other. Since r is arbitrary, η is smooth along the leaves.

In addition to the Sobolev type spaces $L^{2,r}_{\mathscr{F}}$ for r a nonnegative integer, we can use the standard "negative norm" construction to construct the spaces $L^{2,-r}_{\mathscr{F}}$ [14] which are dual to $L^{2,r}_{\mathscr{F}}$. The standard argument which gives the usual Hodge decomposition of the Laplace operator now yields the following result. (Recall once again the basic point that $\Delta_{\mathscr{F}}$ is only elliptic along the leaves of $\mathscr{F}$.)

Proposition 3.5 (*Formal Hodge decomposition*). *For each p, let*

$$\Delta_{\mathscr{F}}\colon L^2(B_M; \mathrm{Alt}^p(T\mathscr{F}_M, B_\rho)) \to L^{2,-2}(B_M; \mathrm{Alt}^p(T\mathscr{F}_M, B_\rho)),$$
$$d_{\mathscr{F}}\colon L^{2,1}_{\mathscr{F}}(B_M; \mathrm{Alt}^{p-1}(T\mathscr{F}_M, B_\rho)) \to L^2(B_M; \mathrm{Alt}^p(T\mathscr{F}_M, B_\rho)),$$
$$d^*_{\mathscr{F}}\colon L^{2,1}_{\mathscr{F}}(B_M; \mathrm{Alt}^{p+1}(T\mathscr{F}_M, B_\rho)) \to L^2(B_M; \mathrm{Alt}^p(T\mathscr{F}_M, B_\rho)).$$

If these maps are all continuous, then we have the following:

(i) $L^2(B_M; \mathrm{Alt}^p(T\mathscr{F}_M, B_\rho)) = \overline{\mathrm{im}(d_{\mathscr{F}})} \oplus \ker\Delta_{\mathscr{F}} \oplus \overline{\mathrm{im}(d^*_{\mathscr{F}})}$.

(ii) *If ω is smooth and $d_{\mathscr{F}}\omega = 0$, then $\omega \in \overline{\mathrm{im}(d_{\mathscr{F}})} \oplus \ker\Delta_{\mathscr{F}}$.*

(iii) *Every element of* $\ker\Delta_{\mathscr{F}}$ *is smooth along the leaves.*

For $p = 1$, we obtain:

Corollary 3.6. *Suppose that for $p = 1$ we have* $\ker\Delta_{\mathscr{F}} = 0$ *and* $\mathrm{im}(d_{\mathscr{F}})$ *is closed. Then the map $H^1(\Gamma, C^\infty(M; V)) \to H^1(\Gamma, L^2(M, V)$ is 0.*

Proof. This follows from Lemmas 2.4, 3.4, and 3.5.

4. Application of the Matsushima-Murakami computations

We recall that under the isomorphism

$$\Psi\colon C^\infty(M \times X; \mathrm{Alt}^*(p_X^* TX, V))^\Gamma \to C^\infty(I_M; \mathrm{Alt}^*(T\mathscr{O}, V))_0^K,\ \Psi(\eta) = \eta^0,$$

given in §2, we have that the exterior derivative in the X-direction d_η corresponds to $(D + d_\rho)(\eta^0)$ (Lemma 2.5). Under the isomorphism of the domain of Ψ with $C^\infty(B_M; \mathrm{Alt}^*(T\mathscr{F}_M, B_\rho))$, the action of $\Delta_{\mathscr{F}}$ on the latter space simply corresponds to the ordinary Laplacian on V-valued forms on X applied in the X-direction on dom(Ψ). Following the computations of [8, pp. 41–44] essentially verbatim, we obtain the following.

Lemma 4.1. *Under the isomorphism* Ψ, $\Delta_{\mathscr{F}}(\eta)$ *corresponds to* $(\Delta_D + \Delta_\rho)(\eta_0)$, *where* Δ_D, Δ_ρ *are differential operators on* $C^\infty(I_M; \mathrm{Alt}^*(T\mathscr{O}, V))_0^K$ *such that the following hold*:

(i) $(\Delta_D \eta^0, \eta^0) \geq 0$, $(\Delta_\rho \eta^0, \eta^0) \geq 0$ *for all* η^0, *where* (,) *denotes the ordinary* L^2 *inner product.*

(ii) *There is a linear operator* $H_\rho = \sum_p^{\oplus} H_\rho^p$ *on* $\mathrm{Alt}^*(\mathfrak{p}, V)$ *such that for any* $y \in I_M$,

$$(\Delta_\rho(\eta^0))(y) = H_\rho(\eta^0(y)).$$

(*Recall the identifications made preceding Lemma* 2.5.) *In particular,* Δ_ρ *is of order* 0.

(iii) [8, *formula 5.12*] *If* $\eta \in C^\infty(B_M; B_\rho)$ (*i.e., is a* 0*-form*), *then*

$$(\Delta_{\mathscr{F}}\eta, \eta) = \int_{\Gamma\backslash G} \left(\sum_{j=1}^N \|X_j \eta^0\|_V^2 + \|\rho(X_j)\eta^0\|_V^2 \right).$$

Here the X_j *are as in the discussion preceding Lemma* 2.5; $\|\ \|_V$ *is simply the norm on* V *deriving from the inner product on* V.

We also recall here the fundamental result of Raghunathan [9], generalizing earlier work of Weil [13].

Lemma 4.2(*Raghunathan* [9]). *For* $p = 1$, ρ *nontrivial and irreducible and for* $\mathfrak{g}$ *with no simple factors isomorphic to any* $\mathrm{so}(1, n)$ *or* $\mathrm{su}(1, n)$, H_ρ^1 *is a positive definite* (*symmetric*) *operator on* $\mathrm{Hom}(\mathfrak{p}, V)$.

Corollary 4.3. *Assume the hypotheses of Lemma* 4.2. *Then* $\ker \Delta_{\mathscr{F}} = (0)$, *and* $\mathrm{im}\, d_{\mathscr{F}}$ *is closed, where these maps are taken as in Proposition* 3.5.

Proof. For any $\eta \in L^2(B_M; \mathrm{Hom}(T\mathscr{F}_M, B_\rho))$ which is smooth along the leaves of $\mathscr{F}_M$, we have

$$(\Delta_{\mathscr{F}}\eta, \eta) \geq (\Delta_\rho \eta^0, \eta^0) = \int_{\Gamma\backslash G} \langle H_\rho^1(\eta^0(y)), \eta^0(y)\rangle\, dy$$
$$\geq c \int \langle \eta^0(y), \eta^0(y)\rangle\, dy \geq c\|\eta^0\|_2.$$

Thus, $\ker \Delta_{\mathscr{F}} = 0$. We now claim that $\mathrm{im}\, d_{\mathscr{F}}$, $d_{\mathscr{F}}\colon L_{\mathscr{F}}^{2,1}(B_M; B_\rho) \to L^2(B_M; \Lambda^1(T\mathscr{F}, B_\rho))$, is closed.

Since $\Delta_{\mathscr{F}} = d_{\mathscr{F}} d_{\mathscr{F}}^* + d_{\mathscr{F}}^* d_{\mathscr{F}}$, if $\eta \in C^\infty(B_M, B_\rho)$ we have $(\Delta_{\mathscr{F}}\eta, \eta) = \|d_{\mathscr{F}}\eta\|_2^2$. It suffices to see that $\|d_{\mathscr{F}}\eta\|_2^2 \geq c\|\eta\|_{2,1,\mathscr{F}}^2 = c((I + \Delta_{\mathscr{F}})\eta, \eta)$ (for some constant c). Thus it suffices to see $\|d_{\mathscr{F}}\eta\|_2^2 \geq c\|\eta\|_2^2$. From Lemma 4.1(iii), we have $\|d_{\mathscr{F}}\eta\|_2^2 = (\Delta_{\mathscr{F}}\eta, \eta) \geq \int \sum_j \|\rho(X_j)\eta^0\|_V^2$. Since ρ contains no G-invariant vectors, any vector $v \in V$ for which $\rho(X_j)v = 0$ for all j, $1 \leq j \leq N$, must be 0. Namely, $\rho(X_j)v = 0$ for all j implies $(\rho|\mathfrak{p})(v) = 0$. Since $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$, we also have $(\rho|\mathfrak{k})(v) = 0$, and hence $\rho(A)v = 0$ for all $A \in \mathfrak{g}$, showing $v = 0$. Thus for $v \neq 0$, $v \to (\sum \|\rho(X_j)v\|^2)/\|v\|^2$ is constant on lines and nonvanishing. Let c be its minimum; we have $\int \sum_j \|\rho(X_j)\eta^0\|_V^2 \geq c \int \|\eta^0\|_V^2$, and this completes the proof.

We can summarize our discussion to this point as follows.

Theorem 4.4. *Let G be a connected semisimple Lie group, $\Gamma \subset G$ a cocompact lattice, and $\rho\colon G \to \mathrm{GL}(V)$ a finite dimensional representation of G not containing the identity. Let M be a compact manifold on which Γ acts smoothly, preserving a smooth volume density. Let Γ act on functions from M into V by translation in M, and ρ in V. Finally assume G has no simple factors locally isomorphic to* $\mathrm{O}(1,n)$ *or* $\mathrm{U}(1,n)$. *Then $H^1(\Gamma; C^\infty(M,V)) \to H^1(\Gamma; L^2(M,V))$ is* 0.

Proof. If Γ is torsion free, this theorem follows from 3.6, 4.2, and 4.3. If Γ is not torsion free, we can choose a torsion free subgroup Γ' of finite index. Applying the theorem for Γ', we deduce that for an action of Γ on $L^2(M,V)$ by affine transformations, there is a Γ' fixed point. This is then a point with a finite Γ-orbit, and the average of this orbit will be a Γ-fixed point, establishing triviality of the corresponding cohomology class.

When ρ contains invariant vectors, Theorem 4.4 is still true. To see this, it suffices to consider the case in which ρ is the trivial representation. Then $L^2(M,V) = L^2(M)$ is a unitary Γ-module, and since Γ has Kazhdan's property, $H^1(\Gamma, L^2(M)) = 0$ [11]. Thus we have:

Theorem 4.5. *Theorem* 4.4 *is true for any finite dimensional representation of G.*

Suppose now that $\phi\colon G \to H$ is a homomorphism where H is a Lie group, and that $\Lambda \subset H$ is a cocompact lattice. Let M be the Γ-space $M = H/\Lambda$. Then TM can be naturally identified with $M \times \mathfrak{h}$, and the H-action is given by translation on M and Ad on $\mathfrak{h}$. Applying Theorem 4.5 to M and the representation $\rho = \mathrm{Ad}_H \circ \phi$, we obtain:

Theorem 4.6. *Let G, Γ be as in Theorem* 4.4, *H a Lie group, $\phi\colon G \to H$ a smooth homomorphism, and $\Lambda \subset H$ a cocompact lattice. Then the Γ action on $M = H/\Lambda$ is L^2-infinitesimally rigid.*

5. Infinitesimal rigidity

We continue with the hypotheses of Theorem 4.6 and pursue the question as to when the conclusion of Theorem 4.6 can be strengthened to assert infinitesimal rigidity. As a G-module, we have $C^\infty(M; TM) \cong C^\infty(M) \otimes \mathfrak{h}$. For any vector bundle $E \to M$, let $J^k E \to M$ be the kth jet bundle. Then we have (using an invariant connection) an isomorphism of G-modules

$$C^\infty(M; J^k(TM)) \cong C^\infty(M) \otimes \sum_{j=0}^{k} (S^j(\mathfrak{h})^* \otimes \mathfrak{h}),$$

and a similar assertion for $L^2(M; J^k TM)$. From Theorem 4.5, we deduce that for every k the map $H^1(\Gamma, C^\infty(M; J^k TM)) \to H^1(\Gamma, L^2(M, J^k TM))$ is 0. We let $L^{2,k}(M; E)$ denote the kth Sobolev space of sections of E. Then we have a commutative diagram:

$$\begin{array}{ccc} C^\infty(M;E) & \xrightarrow{j^k} & C^\infty(M; J^k E) \\ \downarrow & & \downarrow \\ L^{2,k}(M;E) & \longrightarrow & L^2(M; J^k E) \end{array}$$

Let $E = TM$. We know that the right vertical arrow induces 0 on H^1. Our approach to proving infinitesimal rigidity will be to show that the left vertical arrow induces 0 as well.

Let Q_k^∞ denote the quotient space $C^\infty(M; J^k E)/j^k(C^\infty(M;E))$ and Q_k^2 denote the space $L^2(M; J^k E)/L^{2,k}(M;E)$. There is a natural map $Q_k^\infty \to Q_k^2$ which is injective. We then have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} C^\infty(M;E)^\Gamma & \longrightarrow & C^\infty(M;J^kE)^\Gamma & \xrightarrow{\psi_\infty} & (Q_k^\infty)^\Gamma & \longrightarrow & H^1(\Gamma, C^\infty(M;E)) & \longrightarrow & H^1(\Gamma, C^\infty(M;J^kE)) \\ & & \downarrow & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} \\ & & L^2(M;J^kE)^\Gamma & \xrightarrow{\psi_2} & (Q_k^2)^\Gamma & \longrightarrow & H^1(L^{2,k}(M;E)) & \xrightarrow{\psi} & H^1(\Gamma, L^2(M;J^kE)) \end{array}.$$

We know that $\phi_2 = 0$. We claim $\phi_1 = 0$. To see this, it clearly suffices to see that ψ_2 is surjective.

Lemma 5.1. *ψ_∞ is surjective. If Γ is ergodic on M, then* $\dim(Q_k^\infty)^\Gamma < \infty$.

Proof. The injection $j^k\colon C^\infty(M;E) \to C^\infty(M; J^k E)$ is naturally split by the map induced by the projection $J^k E \to J^0 E \cong E$. The naturality implies that ψ_∞ is surjective.

To see the second assertion, we recall the beginnings of the Spencer resolution [3], [2]. Namely, there are differential operators (called the Spencer operators)

$$D_1\colon C^\infty(M; J^k E) \to C^\infty(M; J^{k-1}E \otimes T^*M),$$
$$D_2\colon C^\infty(M; J^{k-1}E \otimes T^*M) \to C^\infty(M; J^{k-2}E \otimes \Lambda^2(T^*M)$$

such that

$$C^\infty(M;E) \xrightarrow{j^k} C^\infty(M;J^kE) \xrightarrow{D_1} C^\infty(M;J^{k-1}E\otimes T^*M) \xrightarrow{D_2} C^\infty(M;J^{k-2}E\otimes\Lambda^2(T^*M))$$

is exact. In particular, $\operatorname{im}(D_1)$ is closed (in the C^∞-topology), and we can identify Q_k^∞ and $\operatorname{im}(D_1)$ as Γ-modules. Thus, it suffices to see that $C^\infty(M;J^{k-1}E\otimes T^*M)^\Gamma$ is finite dimensional. However, this follows from the following lemma.

Lemma 5.2. *If $E\to M$ is a rank n vector bundle, and a group Γ acts on E by vector bundle automorphisms covering an ergodic action on M, then* $\dim(F(M;E)^\Gamma)<\infty$ *where $F(M;E)$ is the space of measurable sections.*

Proof. We first observe that if $\{\phi_i\}$ is any countable family of measurable invariant sections, then ergodicity implies that the set of points $m\in M$ for which $\{\phi_i(m)\}$ is linearly independent in E_m is either null or conull. Suppose now that $\{\phi_i\}_{i\in I}$ is a family of measurable invariant sections maximal with respect to the property of being linearly independent at almost all m. Then $\operatorname{card}(I)\le n$. If ϕ_0 is any other invariant section, by maximality and our remarks above, we can write $\phi_0(m)=\sum c_i(m)\phi_i(m)$. Since ϕ_i,ϕ_0 are Γ-invariant, so are c_i, which by ergodicity implies c_i are essentially constant. Thus, $\{\phi_i\}$ spans $F(M;E)^\Gamma$ over $\mathbf{R}$.

We now return to consider the case when we have $\phi_1=0$.

Lemma 5.3. *If $\phi_0((Q_k^\infty)^\Gamma)$ is dense in $(Q_k^2)^\Gamma$, then $\phi_1=0$.*

Proof. By Lemma 5.1 we have ϕ_0 and ψ_∞ are surjective, and hence ψ_2 is surjective.

Lemma 5.4. *Suppose that for the homomorphism $\phi\colon G\to H$ we have either:*

(a) $\phi(\Gamma)$ *is dense in H; or*

(b) $H=H_1\times H_2$ *is semisimple, Λ projects densely into H_2, and $\phi(\Gamma)$ projects densely into H_1 and trivially into H_2.*

Then $\phi_0((Q_k^\infty)^\Gamma)$ is dense in $(Q_k^2)^\Gamma$.

Remark. The condition on Λ in (b) (given the condition on Γ) is equivalent to ergodicity of Γ on H/Λ. This holds, of course, if $\Lambda\subset H$ is an irreducible lattice.

Proof. (a) Let $f\in L^2(M;J^kE)$ such that $\psi_2(f)$ is Γ-invariant. Then by continuity $\psi_2(f)$ is also H-invariant. Let π be the representation of H on $L^2(M;J^kE)$. Then π defines in the usual way a representation of $C_c^\infty(H)$ on $L^2(M;J^kE)$, and representations of H and $C_c^\infty(H)$ on the Hilbert space Q_k^2. Since $\psi_2(f)$ is H-invariant, we have for all $\alpha\in C_c^\infty(H)$ that $\pi(\alpha)(\psi_2(f))=\psi_2(f)$. However, since $\alpha\in C_c^\infty(H)$ and H is transitive on M, $\pi(\alpha)f \in C^\infty(M; J^k E)$. Thus, $\psi_2(f) = (\psi_0 \circ \psi_\infty)(\pi(\alpha)f)$, and hence $\phi_0((Q_k^\infty)^\Gamma) = (Q_k^2)^\Gamma$.

(b) Let f be as in (a). Then $\psi(f)$ is also H_1-invariant. Since H_1 is normal in $H_1 \times H_2$, $H_1 \times H_2$ acts on the space $(Q_k^2)^{H_1}$. Choose an approximate identity $\alpha_j \in C_c^\infty(H_1 \times H_2)$. Then $\pi(\alpha_j)(\psi_2(f)) \in (Q_k^2)^{H_1}$ and $\pi(\alpha_j)(\psi_2(f)) \to \psi_2(f)$. However, as in (a), $\pi(\alpha_j)f \in C^\infty(M; J^k E)$ so $\pi(\alpha_j)\psi(f) \in \mathrm{im}(\phi_0)$, and hence $\mathrm{im}(\phi_0)$ is dense.

Corollary 5.5. *Under the hypotheses of Lemma 5.4, for each k we have* $H^1(\Gamma, C^\infty(M; TM)) \to H^1(\Gamma, L^{2,k}(M; TM))$ *is* 0.

Proof. See 5.3 and 5.4.

Theorem 5.6. *Suppose G is a connected semisimple Lie group, $\Gamma \subset G$ is a cocompact lattice, and $\phi\colon G \to H$ is a homomorphism where H is another semisimple Lie group. Assume G has no simple factors locally isomorphic to* $\mathrm{SO}(1,n)$ *or* $\mathrm{SU}(1,n)$. *Let $\Lambda \subset H$ be a cocompact lattice. Assume either:*

(a) *$\phi(\Gamma)$ is dense in H; or*

(b) *$H = H_1 \times H_2$, $\phi(\Gamma) \subset H_1 \times \{e\}$ and is dense in H_1, and Λ projects densely onto H_2.*

Then the Γ action on $M = H/\Lambda$ is infinitesimally rigid.

Proof. Let $c\colon \Gamma \to C^\infty(M, TM)$ be a 1-cocycle. By 5.5, for each k we can find $f_k \in L^{2,k}(M, TM)$ such that $\gamma f_k - f_k = c(\gamma)$. Thus, for any $l > k$, $\gamma f_k - f_k = \gamma f_l - f_l$, so that $f_k - f_l$ is Γ-invariant. To see that f_k is smooth, it therefore suffices to prove the following.

Lemma 5.7. *Let M be a compact manifold on which Γ acts ergodically by volume preserving diffeomorphisms. We assume $\Gamma \subset G$ is a lattice where G is any semisimple Lie group with no compact factors. Let (ρ, V) be a finite dimensional representation of G. Then any measurable Γ-invariant section of $M \times V \to M$ is (essentially) smooth.*

Proof. A measurable Γ-invariant section is of course simply a measurable Γ-map $f\colon M \to V$, where Γ acts on V by ρ. Let $(\rho, V) = \sum^{\oplus}(\rho_i, V_i)$ be the decomposition into G-irreducible subspaces, and $q_i\colon V \to V_i$ be projection. If ρ_i is trivial, then any measurable section is constant by ergodicity, and hence smooth, If V_i is nontrivial and μ is a Γ-invariant smooth measure on M, then $(q_i \circ f)_*\mu$ is a Γ-invariant probability measure on V_i. By [6, §3.2] and the Borel density theorem $(q_i \circ f)_*\mu$ is also G-invariant. Since V_i is irreducible, [6, 3.2.2] implies that $(q_i \circ f)_*\mu$ is supported at the origin, i.e. $q_i \circ f = 0$ (a.e.). Thus, f is smooth.

The proof of Theorem 5.6 shows that under the same hypotheses one has vanishing first cohomology for the Γ-module of smooth sections of any natural vector bundle over M in the sense of [12]. In particular, we have:

Theorem 5.8. *Under the hypotheses of Theorem* 5.6, *we have* $H^1(\Gamma, C^\infty(M)) = 0$.

6. Remarks on the proof

The only point in the proof of Theorem 5.6 that required the special nature of the homomorphism ϕ was the proof that the map ψ_2 is surjective. Thus, we make the following definition. Let M be a compact manifold, $E \to M$ a finite dimensional vector bundle, and assume a group Γ acts by vector bundle automorphisms. Let Q_k be the Hilbert space $L^2(M; J^k E)/L^{2,k}(M; E)$.

Definition 6.1. Call the action of Γ on $E \to M$ k-admissible if the map of Γ-invariants $L^2(M; J^k E)^\Gamma \to (Q_k)^\Gamma$ is surjective. Call the action on M k-admissible if the action on TM is k-admissible.

We then have:

Theorem 6.2. *Let G be a connected semisimple Lie group with finite center, no compact factors, and no simple factors locally isomorphic to* $\mathrm{SO}(1, n)$ *or* $\mathrm{SU}(1, n)$, *and let ϕ: $G \to H$ be a homomorphism into another Lie group. Let $\Gamma \subset G$ and $\Lambda \subset H$ be cocompact lattices, and suppose Γ acts ergodically on H/Λ. If the Γ-action on H/Λ is k-admissible for all k, then the action is infinitesimally rigid.*

The arguments of §5 show that every natural vector bundle is k-admissible for all k, for the actions in Theorem 5.6. As observed in Lemma 5.1, the corresponding map of Γ invariants for smooth sections is always surjective.

7. Relations with the automorphism groups of G-structures

Let G be a connected semisimple Lie group with finite center such that every simple factor of G has real rank at least 2. Let $\Gamma \subset G$ be a lattice subgroup. We consider the implications of the condition that $H^1(\Gamma, C^\infty(M)) = 0$ for actions on a compact manifold. For Γ cocompact and special actions of Γ we have established this property in Theorem 5.8. In this section we show how establishing this property in more generality would lead to a proof of the main conjectures of [20].

Definition 7.1. Let G and Γ be as above. We say that Γ satisfies property (V) if for every smooth action of Γ on a compact manifold M which preserves a probability measure in the smooth measure class we have $H^1(\Gamma, C^\infty(M)) = 0$.

We remark that we do not know that any of the Γ under consideration have property (V). We also recall that from Kazhdan's property, we have $H^1(\Gamma, L^2(M)) = 0$.

We now recall the basic conjecture of [20].

Conjecture [20]. *Let G and Γ be as above. Let M be a compact n-manifold, and $H \subset \mathrm{GL}(n, \mathbf{R})$ a real algebraic group that defines a volume density. Suppose Γ acts smoothly on M preserving an H-structure. Then either:*

(i) *there is a nontrivial Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$; or*
(ii) *there is a smooth Γ-invariant Riemannian metric on M.*

This conjecture implies in particular that there is an integer $n(G)$, readily computed from the Lie algebra of G, such that any volume preserving action of Γ on M^n for $n < n(G)$ is finite, i.e., factors through a finite quotient of Γ (see [20] for a discussion). The conjecture has been established for certain H in [17], [18], [19] (see [20] for a unified treatment). Here we show that a basic technique in [20] establishes the conjecture for all H, for any Γ which has property (V).

Theorem 7.2. *Assume the hypotheses of the conjecture. If Γ has property* (V), *then the conclusion of the conjecture is true.*

Lemma 7.3. *Suppose H is a connected Lie group with a closed normal subgroup $N \subset H$ such that $H/N \cong \mathbf{R}^n$. If Γ has property* (V) *and $P \to M$ is a principal H-bundle on which Γ-acts by vector bundle automorphisms, preserving a probability measure of smooth class on M, then there is a Γ-invariant reduction P to N.*

Proof. Since $P/N \to M$ is a principal $\mathbf{R}^n$-bundle, we can choose a smooth section. Thus, as principal $\mathbf{R}^n$-bundles we can write $P/N = M \times \mathbf{R}^n$. The Γ action on P/N corresponds to an action $\gamma(m, v) = (\gamma m, \alpha(\gamma, m) + v)$ where $\alpha \colon \Gamma \times M \to \mathbf{R}^n$. Viewing α as a map $\alpha \colon \gamma \to C^\infty(M, \mathbf{R}^n)$, α is a cocycle. By property (V), α is trivial in cohomology, so there is a Γ-invariant smooth section of $P/N \to M$.

Proof of Theorem 7.2. Since H is algebraic, there is a simply connected solvable subgroup $S \subset H$ such that H/S is compact. Fix a positive integer k. Let $P^{(k)}(M) \to M$ be the kth order frame bundle of M [5], so that $P^{(k)}(M)$ is a principal $\mathrm{GL}(n, \mathbf{R})^{(k)}$-bundle, where we can write $\mathrm{GL}(n, \mathbf{R})^{(k)} = \mathrm{GL}(n, \mathbf{R}) \ltimes U$, where U is a unipotent group. Let $L = S \ltimes U$, which is thus simply connected and solvable. Since Γ preserves an H-structure on M, there is a Γ-invariant reduction of $P^{(k)}(M) \to M$ to a principal $(H \ltimes U)$-bundle $Q \to M$. Let $\tilde{M} = Q/L$, so that $\tilde{M}$ is compact.

If assertion (i) in the conjecture does not hold, we can apply the measurable superrigidity theorem for cocycles ([6], [20, §4]) to deduce that the measurable cocycle $\Gamma \times M \to H \ltimes U$ defined by the Γ-action on Q is measurably equivalent to a cocycle into a compact subgroup of $H \ltimes U$. It follows that there is a Γ-invariant probability measure of smooth class on $\tilde{M}$. (We note, however, that the invariant measure may not itself be smooth.)

Let $\pi\colon \tilde{M} \to M$ be the projection, and $\pi^*(Q) \to \tilde{M}$ the pullback of Q under π. Then $\pi^*(Q)$ has a Γ-invariant reduction to L, say $Q_1 \to \tilde{M}$. Since $L/[L, L]$ is isomorphic to $\mathbf{R}^n$, Lemma 7.3 implies there is a Γ-invariant reduction of Q_1 to $[L, L]$, and continuing inductively, we can reduce the group to $\{e\}$. Thus, there is a Γ-invariant smooth section of $\pi^*(Q) \to \tilde{M}$. From the argument in the proof of [20, Theorem 6.1], and the compactness of $\tilde{M}$, it follows that for any smooth metric ω on M and any k, $\{\gamma^*\omega\}$ has compact closure in the C^k-topology on metrics. The arguments of [20, §6] then show that Γ preserves a smooth Riemannian metric. This proves Theorem 7.2.

We remark that by a suitable version of Shapiro's lemma [1, p. 282],

$$H^1(\Gamma, C^\infty(M)) \cong H^1_d(G, C^\infty(M')),$$

where M' is the induced G-space $\Gamma\backslash(G \times M)$, and H^*_d denotes differentiable cohomology. If Γ is cocompact, then so is M', and G preserves a probability measure of smooth class on M' if Γ does so on M. By [1, p. 279], we can then also write

$$H^1(\Gamma, C^\infty(M)) \cong H^1(\mathfrak{g}, K; C^\infty(M')).$$

Thus, property (V) for Γ can be deduced from the analogous property for differentiable G-cohomology or $(\mathfrak{g}, K)$-cohomology for smooth functions on G-spaces. However, since K is not in general transitive on M', $C^\infty(M')$ will not be an admissible G- (or $(\mathfrak{g}, K)$-)module, and hence the many results known under that hypothesis cannot be immediately applied.

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